# Solar neutrinos: association with $\nu_1$ and $\nu_2$

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## Abstract

This paper is an extended version of a post by the author at the Stack Exchange Physics Forum, an answer to a question about why $\Delta m^2_{12}$ and $\theta_{12}$ are associated with solar neutrinos given that there are three neutrinos considered in the usual oscillation equations Why $\Delta m^2_{12}$ identified with solar?.

## I. INTRODUCTION

There are historical as well as physical reasons why the smaller neutrino squared mass splitting, $\Delta m^2_{12}$, and the $\theta_{12}$ mixing angle between $\nu_1$ and $\nu_2$ are associated with the solar electron neutrino flux and often offered as the solely relevant parameters in simplified two-neutrino models concerned with the observed flavor change of the solar flux.

## II. HISTORY

In 1930 Pauli proposed the neutrino as a second emitted particle in $\beta$-decay to explain the continuous spectrum of the electron (announcing this suggestion in 1933 more publicly at the Seventh Solvay Conference). Fermi (who had attended that 1933 conference) immediately developed a theory of $\beta$-decay process incorporating the neutrino and notably making the first application to fermions of the "Dirac-Jordan-Klein method of second quantization" (to quote from the English translation of the 1934 version of his paper [1]) where the electron and neutrino probability amplitudes $\psi$ and $\varphi$ (and their complex conjugates) became field operators summing annihilation and creation operators acting on occupation numbers for the states. Most people do not realize how important this 1934 paper by Fermi was in this regard.

C. N. Yang (Nobel Prize in Physics 1957 with T.D. Lee for development of the theory that parity is violated in the weak interaction), relates a conversation he had with Eugene Wigner in the cafeteria of Rockefeller University c. 1970. Yang asked Wigner what he thought was Fermi's greatest contribution to physics and was surprised when Wigner told him it was the beta-decay theory. Yang was surprised because he knew that the original Fermi theory had been superceded by electroweak theory with exchange of massive bosons $W^{\pm}$ and $Z$. Wigner explained that he and von Neumann had been thinking about $\beta$-decay for a long time but simply did not know how to create an electron in a nucleus. Yang reminded Wigner that it was he and Jordan who had invented the second quantized $\psi$. Wigner replied, "Yes, yes. *But we never dreamed that it could be used in real physics.*" [2]

In 1934 Yukawa proposed what came to be known as the meson ("middleweight", with mass above lightweight leptons but below heavyweight baryons) as the quantum of the field (the strong force) holding the nucleus protons and neutrons together (despite the intense

Coulomb repulsion between the protons). By 1937 Anderson and other experimenters identified likely candidates for Yukawa's meson in cosmic ray particles. There were inconsistent experimental results for a period (and an intervening world war), but in 1947 Powell discovered that there are two particles involved in the cosmic rays detected, i.e., the $\pi$ or pion (a meson with zero spin) and the $\mu$ or muon (a lepton with fermion spin 1/2) produced when the pion decays (substantially the theoretical proposal made by Marshak and Bethe the same year, and an unknown at the time similar 1943 idea of Sakata and Inoue delivered to a Japanese "meson club" meeting during World War II).[3]

It was realized fairly quickly that the neutral low mass (or massless) particle accompanying the muon in the decay of pions was probably a neutrino $\pi^{\pm} \to \mu^{\pm} + \nu$ and that in the subsequent decay of the muon another two neutrinos were involved $\mu^{\pm} \to e^{\pm} + 2\nu$. Some of the c. 1949 enhanced photographic emulsions from Bristol show plainly visible a long muon track leaving the short track of a pion at approximately a right angle and, ending at various angled tracks, the electron emitted in the muon decay.

It was natural and economical to assume that these neutrinos were the same neutrino believed to accompany the electron in beta decay $n \to p + e^- + \nu$ (we now know it is an antineutrino $\bar{\nu}_e$ emitted in $\beta$-decay).

In 1951 Ray Davis began experimenting with a radiochemical neutrino detection method suggested by Pontecorvo (1946), i.e., ${}^{37}\mathrm{Cl} + \nu_e \to {}^{37}\mathrm{Ar} + e^-$, or $n + \nu_e \to p + e^-$ where one of the neutrons in ${}^{37}_{17}\mathrm{Cl}$ captures a neutrino and transmutes to a proton, making radioactive ${}^{37}_{18}\mathrm{Ar}$, which is extracted later and its decays (predominantly $K$-orbital electron capture with Auger electron ejection) counted to estimate the number of neutrino reactions that occurred on the chlorine. Davis published negative results in 1955, and in 1958 again published failure to detect this reaction, this time using a much larger reactor source of antineutrinos (Savannah River). The failure to detect the reactor *antineutrinos* (but Davis did detect solar *neutrinos* with the same reaction beginning in 1967 at the Homestake Mine site, eventually earning him a Nobel Prize in 2002) with this reaction demonstrated that the neutrino was not its own antiparticle, i.e., $\nu$ and $\bar{\nu}$ were distinct particles (it is still possible that neutrinos are Majorana fermions, i.e., their own antiparticles, but their behavior will be the same as a Dirac fermion in most experiments).[4] This result could be explained by a simple rule for lepton number conservation (a global symmetry) introduced in 1953 by Konopinski and Mahmoud [5], i.e., assign a positive lepton number $L = +1$ to the electron, muon and neutrino, and a negative lepton number $L = -1$ to their antiparticles (this concept has evolved since such that lepton and antilepton number is conserved separately for each flavor): the ${}^{37}\mathrm{Cl} + \bar{\nu}_e \to {}^{37}\mathrm{Ar} + e^-$ reaction is then forbidden since it has $L = -1$ on the left side and $L = +1$ on the right.

In 1956 Reines and Cowan announced detection of the antineutrino in the inverse beta decay process $\bar{\nu} + p \to n + e^+$ (note $L = -1$ on both sides of the equation by the scheme above), also using the Savannah River reactor source, but utilizing a Cd-doped water target (the water provided the protons $p$) sandwiched between liquid scintillator detectors, detecting the prompt double gamma ray annihilation of the positron and a delayed second signal as the neutron $n$ produced in the decay was thermalized (slowed down by elastic collisions with protons in the water) and captured by cadmium, which emits $\sim 9\,\mathrm{MeV}$ in gamma rays in the neutron capture process. A nucleus is excited to $\sim 8\,\mathrm{MeV}$ or more by neutron capture, plus any additional energy the neutron brings from its momentum, though that is generally low for a thermalized neutron that is able to be captured rather than scattered; a photon $\gamma$ or cascade of photons is subsequently emitted to take the excited nucleus back to ground state.[6]

By 1959 there was some question about the nature of the neutrinos in muon decay (and the neutrino in the preceding pion decay) because experimenters could not observe the decay $\mu \to e\gamma$, i.e., if there was only a single kind of neutrino $\nu$, they should also see muons decaying into an electron and a gamma ray [7]. Pontecorvo suggested in 1959 that it might be the case that the muon and electron neutrinos are distinct particles $\nu_\mu \neq \nu_e$. In 1962 Lederman, Schwartz and Steinberger observed 24 instances of $\bar{\nu}_\mu + p \to \mu^+ + n$, but none of $\bar{\nu}_\mu + p \to e^+ + n$, which should have been equally common in their experiment if $\bar{\nu}_e$ and $\bar{\nu}_\mu$ were the same particle.[5] At this point in the timeline we now have two flavors of neutrino.

A few years prior, in 1955, Gell-Mann and Pais published a paper describing the neutral kaon system of $K^0$ and $\bar{K}^0$ (they denoted these $\theta^0$ and $\bar{\theta}^0$ at that time) as a particle mixture, i.e., superpositions of underlying mass states in today's notation $K_S$ or "K-short" and $K_L$ or "K-long", or $K_1$ and $K_2$ if CP is conserved (1955 notation $\theta_1^0$ and $\theta_2^0$) with short and long decay times to pions respectively (or write $K_S$ and $K_L$ in terms of $K^0$ and $\bar{K}^0$ to describe strangeness oscillations).[8] [9] In 1957 Pontecorvo [10] considered whether there might be other neutral particles differing from their antiparticles for which particle→antiparticle transitions are not forbidden. He looked at mesonium, a bound system of $\mu^+e^-$ (an "atom" made of a heavy positive muon and an electron, sometimes called the "perfect atom" by researchers today, being composed of point-like leptons and so free of perturbations arising from nuclear size effects) possible inversions to antimesonium ($\mu^- e^+$), but added a couple of lines at the end of the article wondering whether a neutrino might transform into an antineutrino, which would permit $\nu \leftrightarrow \bar{\nu}$ transitions in vacuum. This was 1957 so he could not know yet that the 1962 Lederman experiment mentioned above would establish that there were two neutrinos, one associated with the electron and one associated with the muon (or he might have suggested those transitions rather than particle-antiparticle), however, this was one of the first suggestions that neutrinos might change their nature in flight.

Even more interesting was the 1962 paper by Maki *et al* [11] that tried to salvage a now obscure theory of $B^+$ matter that was in jeopardy because of the Lederman group discovery that the muon and electron neutrinos were distinct particles (recent experimental results with the $K^0$ were also a problem). Maki and his collaborators proposed that the "true neutrinos" are linear combinations of the *weak neutrinos* $\nu_e$ and $\nu_\mu$ that interact with their respective leptons $e$ and $\mu$. They described for the first time neutrino *weak interaction states as combinations of mass states governed by a mixing angle* $\delta$ in their equation (2.18):

$$\nu_e = \nu_1 \cos\delta - \nu_2 \sin\delta$$
$$\nu_\mu = \nu_1 \sin\delta + \nu_2 \cos\delta$$

Maki *et al* added that these weak neutrinos were not stable and could transmute $\nu_e \leftrightarrow \nu_\mu$.

Having established the context, we find Ray Davis in 1968 [4] publishing first results from the Homestake Mine detector using the ${}^{37}\mathrm{Cl} + \nu_e \to {}^{37}\mathrm{Ar} + e^-$ reaction he had worked with in the 1950's. Davis reported an upper limit on the detected solar electron neutrino flux as 3 SNU. The SNU, Solar Neutrino Unit, is defined by $\mathrm{SNU} \equiv 10^{-36}$ *interactions per target atom per second.* The threshold of the ${}^{37}\mathrm{Cl} + \nu_e \to {}^{37}\mathrm{Ar} + e^-$ reaction is is 0.81 MeV, so the expected flux could not contain any of the predominant neutrino flux of the Sun, the $p + p \to {}^2\mathrm{H} + e^+ + \nu_e$ fusion with maximum neutrino energy of 0.423 MeV.

The solar neutrinos with sufficient energy for the ${}^{37}\mathrm{Cl}$ capture, primarily ${}^8\mathrm{B}$ neutrinos at $\langle E_\nu \rangle = 6.735$ MeV, but including ${}^7\mathrm{Be}$ and a few other species, were predicted by John Bahcall (within the SSM Standard Solar Model he helped construct [12]) to provide $7.5 \pm 3$ SNU detected flux at Homestake. Thus began the "solar neutrino problem," the solution of which over the next three decades required development of the MSW effect theory (Wolfenstein

1978, 1986 Mikheyev and Smirnov [13]) and the results of the SNO experiment 2001.

SNO, reporting data collected 2 November 1999 through 28 May 2001 (their Phase-I campaign), separately detected solar $\nu_e$ flux (kinetic energy threshold 5 MeV) as $\phi_e = 1.76 \times 10^6\,\mathrm{cm}^{-2}\,\mathrm{s}^{-1}$ via charged current CC reaction $\nu_e + d \to p + p + e^-$ where $d$ is deuterium ${}^2_1\mathrm{H}$).[14]

The total flux measured with the neutral current NC reaction $\nu_x + d \to n + p + \nu_x$ (breakup of the deuterium nucleus by neutrino) where $\nu_x$ includes all neutrino types was $\phi_{\nu_e,\nu_\mu,\nu_\tau} = 5.09 \times 10^6\,\mathrm{cm}^{-2}\,\mathrm{s}^{-1}$ (the threshold for this reaction is 2.2 MeV). By detecting solar neutrinos of all flavors with the NC reaction, SNO verified that the expected amount of SSM neutrino flux for ${}^8\mathrm{B}$ was present ($\phi_{SSM} = 5.05 \times 10^6\,\mathrm{cm}^{-2}\,\mathrm{s}^{-1}$ from Bahcall's 2001 model), but that only about a third of the initial $\nu_e$ at production in the Sun remained in the detected flux (detected by the CC reaction), i.e., the $\nu_e$ had somehow become $\nu_\mu$ (or $\nu_\tau$).

The Davis 1968 experiment (the result of which was first confirmed by the Kamiokande [15] imaging water Cherenkov detector in 1989 observing neutrino arrival time, direction and energy in neutrino-electron scattering $\nu_e e^- \to \nu_e e^-$) resulted in many proposals for a solution to the deficit in solar electron neutrino $\nu_e$ flux, one of which was the *suggestion that neutrinos oscillate between flavors*, 1969 Gribov and Pontecorvo.[16] They arrived at the mixing relations (similar to 1962 Maki *et al*)

$$\nu_e = \cos(\theta)\,\nu_1 + \sin(\theta)\,\nu_2\,, \qquad \nu_\mu = -\sin(\theta)\,\nu_1 + \cos(\theta)\,\nu_2$$

(The negative sign is placed on the $\nu_1$ term of $\nu_\mu$ here, while 1962 Maki *et al* given above placed it on the $\nu_2$ term of $\nu_e$. It is simply an intrinsic phase $e^{i\pi}$ assuring the two flavor states are orthogonal and can be placed on $\nu_e$ or $\nu_\mu$ or a combination as long as the difference is $\pi$. [17]) Gribov and Pontecorvo were proposing that the $\nu_1, \nu_2$ were fields of Majorana neutrinos with masses $m_1$ and $m_2$, so most of their scheme is not directly recognizable in the Dirac form we normally see.

Reines wrote in 1979 [18] that the number of neutrino types and the particular mixing scheme could be associated with the specific reduction in expected neutrino flux at sufficient distance. For example, that

> ...if there are three neutrino types, $\nu_e, \nu_\mu, \nu_\tau$ with maximal mixing between them then for a great enough distance from the source and starting with $\nu_\mu$ only we

would find

$$\nu_\mu \to 1/3\nu_e, 1/3\nu_\tau, 1/3\nu_\mu$$

The concept of neutrino mixing was well enough known that by 1977 we had Wolfenstein writing incidentally in an article [19] proposing a matter potential for neutrinos (publishing in 1978, his idea was subsequently used to create the MSW solution to the solar neutrino deficit by Smirnov and Mikheyev in 1986 as we mentioned above):

> Considering the case of $\nu_e$ and $\nu_\mu$, vacuum oscillations require that the eigenstates in vacuum are mixtures
>
> $$\begin{aligned} |\nu_1\rangle &= |\nu_e\rangle \cos\theta_v - |\nu_\mu\rangle \sin\theta_v \\ |\nu_2\rangle &= |\nu_e\rangle \sin\theta_v + |\nu_\mu\rangle \cos\theta_v \end{aligned} \tag{1}$$
>
> with distinct masses $m_1$ and $m_2$ ($m_1 > m_2$). Neutrino oscillations result from the difference in the phase factors governing the time dependence of $\nu_1$ and $\nu_2$,
>
> $$|\nu_i t\rangle \sim \exp(-itm_i^2/2k)$$
>
> The characteristic oscillation length in the vacuum is $l_v(k) = 4\pi k/(m_1^2 - m_2^2)$.

We note that $k$ in the above is the neutrino momentum, or energy given that the neutrino is always ultrarelativistic.

We find Sciama writing similarly in 1981 [20]:

> $$\underbrace{\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix}}_{\text{Weak interaction eigenstates}} = \underbrace{\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}}_{\text{Mixing matrix}} \underbrace{\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}}_{\text{Mass eigenstates}} \tag{2}$$
>
> The mixing matrix is orthogonal and so defines a mixing angle $\alpha$. The probability $P$ for a certain neutrino, say $\nu_e$, of energy $E$ to turn into $\nu_\mu$ after propagating a distance $L$ is then given by

$$P = \sin^2 2\alpha \sin^2 \frac{(m_1^2 - m_2^2)\, L}{4E} \quad (3)$$

Thus a detector sensitive only to $\nu_e$ could measure $m_1^2 - m_2^2$, the difference of the squared masses of the propagating eigenstates. If several flavors are involved (the $\tau$ lepton was discovered in 1978 at SLAC so it was highly likely there would be a third neutrino flavor) the mixing matrix is larger and the situation more complicated.

Thus far then, we see the mass splitting $\Delta m_{12}^2$ and mixing angle $\theta_{12}$ is associated with electron neutrinos. Why are electron neutrinos associated with the Sun?

## III. WHY SOLAR ELECTRON NEUTRINOS

The proton-proton cycle (dominant in lower main sequence stars like the Sun) [21] consists of several fusion paths creating proton-rich nuclei that lie above the band of nuclear stability (if you plot nuclide number of protons on ordinate and number of neutrons on abscissa, the valley of stability is approximately a 45° line going up left to right for light nuclei, i.e., equal number of protons and neutrons in the nuclide $N = Z = A/2$, but begins to droop for heavy nuclei as more neutrons are required to overcome the mutual Coulomb repulsion between many protons) [22].

These nuclei decrease their unstable p:n ratio through $\beta^+$-decay, e.g., $p+p \rightarrow {}_1^2\mathrm{H}+e^+ +\nu_e$ (${}_1^2\mathrm{H}$ being deuterium, composed of one proton and one neutron, resulting from $\beta^+$ decay of a diproton $pp$). There is not enough energy available in the solar core to create a $\pi$ meson in $p \rightarrow n+\pi^+$ reaction via the strong force, so the weak interaction mediates the final state, producing leptons necessarily. There is not enough energy available in these weak decays (for example, $pp$ $E_\nu^{max} = 423\,\mathrm{keV}$, $^8\mathrm{B}$ $E_\nu^{max} = 16.3\,\mathrm{MeV}$) to produce $\mu^+$ (mass $\sim 105\,\mathrm{MeV}$) or $\tau^+$ (mass $\sim 1.7\,\mathrm{GeV}$), so the produced leptons are all $e^+ + \nu_e$.[23] This is why electron neutrinos are associated with the Sun.

## IV. DERIVE TWO-NEUTRINO OSCILLATION EQUATION

It is worthwhile to take a moment and examine in detail at how the oscillation transition equation above is typically produced [1]. The mass eigenstates of the neutrino can be modelled as free particle solutions to the wave equation with neutrinos propagating as plane waves (the neutrinos propagate as wave packets, but that somewhat more complicated approach results in the same transition equation):

$$|\nu_j(t)\rangle = |\nu_j\rangle e^{-ip_j(x_d - x_s)} \tag{4}$$

In the context of relativistic quantum mechanics we see then

$$e^{-ip_j(x_d - x_s)} = e^{-i(E_jT - p_jL)} = e^{-i\phi_j} \tag{5}$$

In that expression, $p_j$ are the momentum four-vectors of the neutrino mass states. Properly denoted they would be $p^\mu = \{E, p_x, p_y, p_z\}$ (contravariant four-vector). Similarly, the position four-vectors of the neutrino at $d$ destination and $s$ source $x_d, x_s$ would be formally denoted $x^\mu = \{t, x_x, x_y, x_z\}$ (we are trying to be clear here, so labelled the three Cartesian components $x, y, z$; the four-vector components are normally indexed $\mu = 0, 1, 2, 3$) The scalar product of the momentum and position four-vectors then gives the phase $\phi_{j,k}$ of each neutrino mass state (more on the phase below), $\phi_{j,k} = x^\mu p_\mu = Et - \vec{p} \cdot \vec{x} = Et - \vec{p}L$ in the argument of the exponential ($\vec{p}$ denoting the 3-vector component extracted in the scalar product).

Now we recall (we are using the two-neutrino mixing matrix shown in Eq. (2) above) that the interaction neutrino (the flavor in a weak interaction) for an electron neutrino is

$$\nu_e = \cos\theta\, \nu_1 + \sin\theta\, \nu_2 \tag{6}$$

and that the mass states $\nu_{1,2}$ can be thought of in terms of flavor $e, \mu$ inverting the mixing relations:

$$\begin{aligned} \nu_1 &= \cos\theta\, \nu_e - \sin\theta\, \nu_\mu \\ \nu_2 &= \sin\theta\, \nu_e + \cos\theta\, \nu_\mu \end{aligned} \tag{7}$$

To calculate the transition probability of an electron neutrino equation, we can project out the $\nu_\mu$ flavor content of the initial $\nu_e$ propagation state ket $|\nu_e(L,T)\rangle$ and square:

[1] The following sources were consulted throughout this derivation: [24] [25].

$$P(\nu_e \to \nu_\mu) = |\langle \nu_\mu \mid \nu_e(L,T)\rangle|^2$$

To accomplish that simply, we rewrite the equation for the electron neutrino ket at production, replacing the mass states $\nu_1, \nu_2$ with their flavor content expressions (notice that if the mixing angle $\theta_{12}$ were zero, then the $\nu_e$ neutrino would be created as a single mass state $\nu_1$ in vacuum):

$$\begin{aligned}
|\nu_e(L,T)\rangle &= (\cos\theta)\ \underbrace{|\nu_1\rangle}_{\text{replace}}\ e^{-i\phi_1} + (\sin\theta)\ \underbrace{|\nu_2\rangle}_{\text{replace}}\ e^{-i\phi_2} \\
&= (\cos\theta)\ \underbrace{(\cos\theta|\nu_e\rangle - \sin\theta|\nu_\mu\rangle)}_{\text{flavor content of mass state 1}} e^{-i\phi_1} + (\sin\theta)\ \underbrace{(\sin\theta|\nu_e\rangle + \cos\theta|\nu_\mu\rangle)}_{\text{... of mass state 2}} e^{-i\phi_2}
\end{aligned}$$

We then rearrange that equation so we can distinguish the $|\nu_e\rangle$ and $|\nu_\mu\rangle$ factors, carrying out trigonometric term products also:

$$|\nu_e(L,T)\rangle = |\nu_e\rangle \left[\cos^2\theta e^{-\phi_1} + \sin^2\theta e^{-\phi_2}\right] + |\nu_\mu\rangle \left[\sin\theta\cos\theta \left(e^{-i\phi_2} - e^{-i\phi_1}\right)\right]$$

Only the $|\nu_\mu\rangle$ term from the production ket $|\nu_e(L,T)\rangle$ expression above will survive the inner product with the $\langle\nu_\mu|$ bra. The flavor basis states $\alpha, \beta \in \{e, \mu\}$ are orthogonal so their inner product is zero if the flavors are not identical, i.e., $\langle \nu_\beta \mid \nu_\alpha\rangle = \delta_{\alpha\beta}$ where $\delta_{\alpha\beta}$ is the Kronecker delta ($\delta_{\alpha\beta} = 1$ for $\alpha = \beta$, zero if $\alpha \neq \beta$). We therefore have:

$$\begin{aligned}
P(\nu_e \to \nu_\mu) &= |\langle \nu_\mu \mid \nu_e(L,T)\rangle|^2 \\
&= \langle \nu_\mu \mid \nu_\mu\rangle \left|\sin\theta\cos\theta \left(e^{-i\phi_2} - e^{-i\phi_1}\right)\right|^2 \\
&= (1)\left(\sin^2\theta\cos^2\theta\right)\left|\left(e^{-i\phi_2} - e^{-i\phi_1}\right)\right|^2
\end{aligned}$$

Using the Euler formula and trig simplification, the squared modulus[2] of the difference of the exponentials simplifies to:

$$\begin{aligned}
\left|\left(e^{-i\phi_2} - e^{-i\phi_1}\right)\right|^2 &= \left(e^{-i\phi_2} - e^{-i\phi_1}\right)\left(e^{+i\phi_2} - e^{+i\phi_1}\right) \\
&= 2\left[1 - \cos\left(\phi_1 - \phi_2\right)\right]
\end{aligned}$$

[2] Sometimes referred to as the field norm, $(\sqrt{zz^*})^2$.

That gives us

$$P(\nu_e \to \nu_\mu) = 2\left(\sin^2\theta\cos^2\theta\right)\left[1 - \cos\left(\phi_1 - \phi_2\right)\right]$$

Simplifying with more trigonometry identities we obtain

$$P(\nu_e \to \nu_\mu) = \sin^2(2\theta)\sin^2\left(\frac{\phi_2 - \phi_1}{2}\right)$$

Notice that if there is no phase difference, i.e., if $\phi_2 = \phi_1$, then there is no probability of flavor oscillation $P(\nu_e \to \nu_\mu) = 0$. Now from above we had the phase of the propagating neutrinos as $\phi_{j,k} = (E_{j,k}T - p_{j,k}L)$. We can then write that the difference in phase is

$$\Delta\phi_{jk} = \phi_j - \phi_k = (E_j - E_k)T - (|p_j| - |p_k|)L$$

The notation $|p_{j,k}|$ is emphasizing that this quantity is the 3-momentum scalar value in the direction of propagation of the neutrino (obtained from the scalar product of position and momentum four-vectors earlier above). You could say that the distance is $L = \vec{k}(x_d - x_s)$ ($x_d$ the detector location, $x_s$ the source), with $\vec{k}$ the unit vector in the direction of neutrino momentum and momentum value $|p_{j,k}| = |\vec{p}_{j,k}|$ or $\vec{p}_{j,k} = \vec{k}|p_{j,k}|$.

We may rewrite $\Delta\phi_{jk}$ above as

$$\Delta\phi_{jk} = (E_j - E_k)T - \left(\tfrac{|p_j|^2 - |p_k|^2}{|p_j| + |p_k|}L\right)$$

That was obtained simply by multiplying $(|p_j| - |p_k|)L$ by $(|p_j| + |p_k|) / (|p_j| + |p_k|)$.

Now let us rewrite each of the squared momenta in the numerator above, $|p_j|^2 - |p_k|^2$, using the relativistic energy relation $E^2 = p^2 + m^2 \implies p^2 = E^2 - m^2$ and the fact that $E_j^2 - E_k^2 = (E_j - E_k)(E_j + E_k)$. We pull that fraction being subtracted above to make it clear what is being done (and the signs resulting):

$$(-)\frac{|p_j|^2 - |p_k|^2}{|p_j| + |p_k|}L = (-)\left[\tfrac{(E_j - E_k)(E_j + E_k)}{|p_j| + |p_k|} - \tfrac{m_j^2 - m_k^2}{|p_j| + |p_k|}\right]L$$

We replace the original term with the expanded term in the $\Delta\phi_{jk}$ equation above:

$$\Delta\phi_{jk} = (E_j - E_k)T - \left(\frac{|p_j|^2 - |p_k|^2}{|p_j| + |p_k|}L\right)$$
$$\Delta\phi_{jk} = (E_j - E_k)T - \left[\frac{(E_j - E_k)(E_j + E_k)}{|p_j| + |p_k|} - \frac{m_j^2 - m_k^2}{|p_j| + |p_k|}\right] L$$

We now pull the mass squared fraction out of the bracket, being careful to apply the two negative signs to obtain a positive term and retain the factor of $L$:

$$\Delta\phi_{jk} = (E_j - E_k)T - \left[\frac{(E_j - E_k)(E_j + E_k)}{|p_j| + |p_k|}\right] L + \frac{m_j^2 - m_k^2}{|p_j| + |p_k|}L$$

Now we see why we factored $E_j^2 - E_k^2 = (E_j - E_k)(E_j + E_k)$ above. It is clear that the time $T$ and the length $L$ both share a factor $(E_j - E_k)$, so let us apply that factorization accordingly in the expression above:

$$\Delta\phi_{jk} = (E_j - E_k)\left[T - \frac{(E_j + E_k)}{|p_j| + |p_k|}L\right] + \frac{m_j^2 - m_k^2}{|p_j| + |p_k|}L$$

So what does all this algebra buy us? The result is that we do not need to make the usual unphysical claims that $E_j = E_k$ or $p_j = p_k$ to obtain a usable form for the phase difference $\Delta\phi_{jk}$, necessary to predict or analyze neutrino oscillation (in general there is no reason whatsoever to assume that the different mass eigenstates composing a flavor neutrino state have either the same energy or the same momentum). It is only necessary to note that the velocity of the neutrino is almost indistinguishable from $c$ in most cases, so $T \simeq L$, since $T = L/c$. In the natural units of high energy physics, $\hbar = c = 1$, mass and momentum are units of $E$ (for example, GeV), time and length are identical units $1/E$.

Making that approximation, i.e., that $T \simeq L$ above, you see $(E_j - E_k)T - (E_j - E_k)L = 0$ null the bracketed expression in the $\Delta\phi_{jk}$ equation above and the phase difference is then simply the remaining non-zero term

$$\Delta\phi_{jk} = \frac{m_j^2 - m_k^2}{|p_j| + |p_k|}L = \frac{\Delta m_{jk}^2}{2p}L$$

where $p = (p_j + p_k)/2$ (and $E$ may replace $p$ in this final term, i.e., we may safely neglect the dependence of $p_j, p_k$ on the small masses $m_j, m_k$ and treat $p$ as the zero neutrino mass momentum $p = E$).

But what about the factor of

$$\frac{(E_j + E_k)}{|p_j| + |p_k|}$$

on $(E_j - E_k)L$ in the bracketed expression? Does that not make the difference $(E_j - E_k)T - (E_j - E_k)L$ nonzero? You can see more clearly that it does not if you replace the momentum terms $p_{j,k}$ in the denominator of that fraction with their energy form using the binomial expansion $(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots$ to expand $p_{j,k} = \left(E_{j,k}^2 - m_{j,k}^2\right)^{1/2}$ to first order:

$$\frac{E_j + E_k}{p_j + p_k} = \frac{E_j + E_k}{E_j\left(1 - \frac{m_j^2}{2E_j^2}\right) + E_k\left(1 - \frac{m_k^2}{2E_k^2}\right)}$$

For a sample case of neutrino energy $E_\nu = 1\,\mathrm{MeV}$ and mass eigenstate $m_{j,k} \approx 0.07\,\mathrm{eV}$ (a guess consistent with observations), the correction for the $T \sim L$ approximation above is then

$$\sim \mathcal{O}\left(\frac{m_{j,k}^2}{2E_{j,k}^2}\right) \sim \mathcal{O}\left(10^{-15}\right)$$

So

$$\frac{E_j + E_k}{E_j\left(1 - 10^{-15}\right) + E_k\left(1 - 10^{-15}\right)} \approx \frac{E_j + E_k}{E_j + E_k} = 1$$

and $(E_j - E_k)T - (E_j - E_k)L = 0$ is valid, given $T \simeq L$. The resulting neutrino relative phase $\Delta\phi_{jk} = \frac{\Delta m_{jk}^2}{2E}L$ (which we note is Lorentz Invariant) is inserted in the oscillation equation from above to obtain the familiar two-neutrino transition expression:

$$\begin{aligned} P(\nu_e \to \nu_\mu) &= \sin^2(2\theta)\sin^2\left(\frac{\phi_2 - \phi_1}{2}\right) \\ &= \sin^2(2\theta)\sin^2\left(\frac{\Delta\phi}{2}\right) \\ &= \sin^2(2\theta)\sin^2\left(\frac{\Delta m_{jk}^2 L}{4E}\right) \end{aligned} \quad (8)$$

Notice the factor of 4 emerges in the denominator of the phase term as a factor of 2 was present from the earlier trigonometric identities and the $\Delta\phi$ expression brought a second factor of 2 ($2E$ specifically). As we discussed above, we are able to treat the neutrino as massless and replace $p$ by $E$ in $\Delta\phi_{jk}$. Flavor oscillation is observable only as the manifestation of the difference in these (squared) minute masses, not their absolute values. If there is no difference in mass, then there is no flavor oscillation (hence the excitement in physics when evidence of neutrino oscillation appeared, i.e., neutrinos must have mass for this to occur, generally speaking).

We note two exceptions to our suggestion that neutrinos are always ultrarelativistic $v \approx c$. The theoretical remnant neutrino background from the Big Bang, the so-called Cosmic Neutrino Background C$\nu$B, includes at least two non-relativistic neutrino mass eigenstates, given their likely temperature today $T_\nu^0 \simeq 1.68 \times 10^{-4}\,\mathrm{eV} \simeq 1.95\,\mathrm{K}$.[26] That is, $k_B T < mc^2$, their temperature (kinetic energy more or less) is lower than the likely neutrino rest mass of two of the three mass states $\mathcal{O}(10^{-3} - 10^{-2}\,\mathrm{eV})$. The neutrino is non-relativistic also in direct mass experiments (e.g. KATRIN) observing the electron energy very close to the $\beta$-decay endpoint in $n \to p + e + \bar{\nu}_e$ where the electron $e$ has almost all of the $\beta$-decay energy and the accompanying neutrino $\bar{\nu}_e$ has almost none (so the neutrino energy in that case is simply its rest mass, the indirect target of the experiment).[27]

## V. NEUTRINO FLAVOR CHANGE AND THE SUN

We have shown above that historically $\nu_1, \nu_2$ and their mass difference $\Delta m_{12}^2$ were associated with $\nu_e$ (and $\bar{\nu}_e$) electron neutrinos, as well as the angle referring to their mixing, $\theta_{12}$. We explained why only electron neutrinos are produced in the Sun, so on this basis it is easy to see the connection among these entities. Why, however, would a single squared mass splitting and a single mixing angle be able to accurately characterize solar neutrino oscillations when there are three neutrinos, three squared mass splittings, three mixing angles, etc.? Before we begin to address that question, we must note that to be precise, *there are not any solar neutrino flavor oscillations*, at least no one on Earth has detected periodic $L/E$ dependendent flavor change of the $\nu_e$ produced in the solar fusion chains. [17]

However, there is flavor change , since fewer solar electron neutrinos $\nu_e$ are detected than predicted by the SSM Standard Solar Model and other explanations have been ruled out. [28] About 0.55 of the expected low energy $\nu_e$ (say below 1 MeV, think *pp* and $^7$Be neutrinos) are detected. On the other hand, only about 0.3 of the higher energy solar neutrinos, primarily $^8$B above 5 MeV, are observed. This is displayed in the following figure Fig. 1 from the 2014 Borexino experiment [29]:

In Fig. 1 the pink (or fuschia perhaps) band is $^8$B survival probability, calculated over

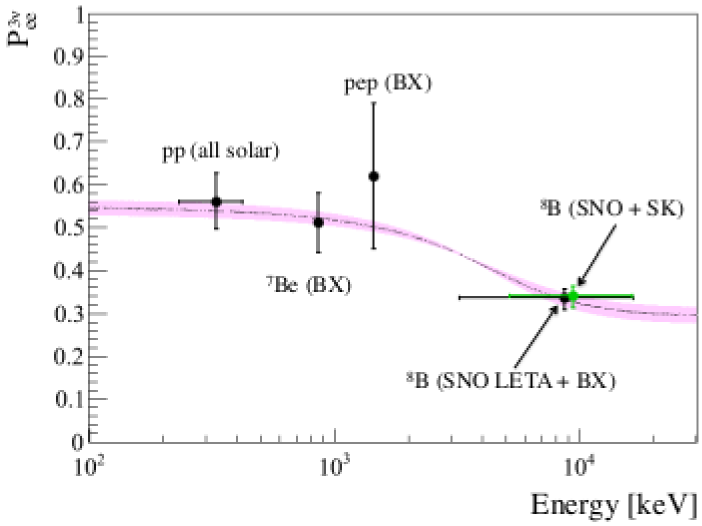


FIG. 1. 2014 Borexino experiment Figure 84. See text for description.

the entire energy range of solar neutrinos using the MSW-LMA model (MSW effect and Large Mixing Angle). The black points represent experimental observations of the particular species (e.g., *pp*, $^{8}$B, etc.), with labels for the experiment. For example, SNO (which we have mentioned already), SK is SuperKamiokande, BX is Borexino. The "*pp* (all solar)" label includes data from all experiments (including Borexino) measuring solar neutrinos at that low energy, e.g. 1997 GALLEX [30] utilizing a radiochemical method similar to the Davis chlorine technique discussed above, using instead the 233 keV threshold $^{71}_{31}$Ga (gallium) neutrino capture reaction, where one of the neutrons in $^{71}_{31}$Ga captures an incoming solar neutrino $\nu_e$ and transmutes to a proton $\nu_e + n \longrightarrow p + e^-$, making radioactive $^{71}_{32}$Ge (germanium), i.e., through the process $^{71}_{31}\text{Ga}(\nu_e, e^-)^{71}_{32}\text{Ge}$. The $^{71}_{32}$Ge is extracted later and a count made of the signal from the Auger electrons and x-rays it produces when it subsequently inverse $\beta$-decays (electron capture to the ground state of $^{71}$Ga) with half-life $\sim$ 11 days (using gas-filled proportional counters designed to produce a single pulse with amplitude and rise-time proportional to the incident $\beta$ particle or x-ray photon energy).

### A. how determine if oscillations are observable

When you consider detecting neutrino flavor change you first need to calculate the distance scale ($\lambda_{osc}$) over which oscillation effects could be observable.

The characteristic wavelength $\lambda_{osc}$, also known as oscillation length $L_{osc}$, can by calculated by setting the oscillation term (phase difference), often denoted $\Delta_{jk}$, (the argument to the $\sin^2$ term in Eq. (8)) equal to $\pi$ and solving for $L$:

$$\begin{aligned} \pi &= \frac{\Delta m_{jk}^2 L}{4E} && \text{set osc factor } \Delta_{jk} \text{ equal to } \pi \\ 4\pi E &= \Delta m_{jk}^2 L && \text{solve for } L \\ \frac{4\pi E}{\Delta m_{jk}^2} &= L_{osc} = \lambda_{osc} && \text{osc wavelength} \end{aligned} \tag{9}$$

see Section 6.4 and Chapter 8 in [31] and, in [24], Chapter 14, in particular §14.7 and Eqs (14.36), (14.39), and (14.50) and surrounding text.

If you apply the correct SI $\hbar c$ factors to natural units Eq. (9) you can obtain a convenient formula for the vacuum oscillation length in km for a given neutrino energy $E_\nu$ in GeV and mass split difference in $\text{eV}^2$:

$$\begin{aligned} \lambda_{osc} &= \frac{4\pi E}{\Delta m_{jk}^2} = \frac{(12.57)\, E\, (\text{GeV})}{\Delta m_{jk}^2\, (\text{eV}^2)} \\ \lambda_{osc}(\text{km}) &= \frac{12.57\, E\ \times 10^9\, (\text{eV})}{\Delta m_{jk}^2\, (\text{eV}^2)} = \frac{12.57\ \times 10^9\, (\hbar c)}{(\text{eV})} \\ & \quad \text{select } \hbar c \text{ above in units (eV km) to cancel denom remaining eV and leave km} \\ \lambda_{osc}(\text{km}) &= \frac{12.57\ \times 10^9\ (\,1.97\ \times 10^{-10})\, [\text{eV km}]\,)}{(\text{eV})} = 2.47\, \frac{E\, (\text{GeV})}{\Delta m_{jk}^2\, (\text{eV}^2)} \end{aligned} \tag{10}$$

For example, the oscillation length is $\lambda_{osc} \sim 8.8\,\text{km}$ for the $\Delta m_{12}^2 = 7.5\ \times 10^{-5}\,\text{eV}^2$ mass split with neutrino energy $267\,\text{keV} = 267\ \times 10^3 \times 10^{-9} = 267\ \times 10^{-6}\,\text{GeV}$:

$$\lambda_{osc} = 2.47\ \times \frac{(267\ \times 10^{-6}\,\text{GeV})}{7.5\ \times 10^{-5}\text{eV}^2} = 8.8\,\text{km}$$

In order to detect oscillations the oscillation length $\lambda_{jk}$ of the squared mass splitting(s) being probed should be close to or slightly less than the baseline $L$ (distance from source to detector) [24]. Clearly, the distance from the Sun to Earth, $L \sim 1.50\ \times 10^8\,\text{km}$ (1 au), is far greater than the $\lambda_{12}$ associated with $\Delta m_{12}^2 = 7.5\ \times 10^{-5}\,\text{eV}^2$. The other mass splitting produces a similarly small oscillation length, $\lambda_{32} \approx \lambda_{31} \sim 0.3\,\text{km}$ for the $\Delta m_{31}^2 = 2.457\ \times 10^{-3}\,\text{eV}$ squared mass splitting.

Put another way, an optimal neutrino oscillation experiment is one in which the ratio of the neutrino energy and baseline are of the same order as the mass splitting, $E/L \sim \Delta m^2$ [31].

You require $\Delta_{ij} = \Delta m_{jk} L/(4E) \gtrsim 1$ for at least one $\Delta m_{jk}^2$, i.e., the neutrino oscillation length should be of the order of, or smaller, than the source-detector distance [24]. If you keep in mind that we are talking about the $\Delta_{ij}$ argument of the second $\sin^2$ term in Eq. (8) [3], it should be obvious that designing the experiment such that this dynamic argument reached $\pi/2 = 1.57\,\text{rad}$ (or a few odd integer multiples of $\pi/2$), for example, would allow the transition probability to reach maximum (and that $\pi$ multiples would result in zero).

For the distance between the Earth and the Sun $L \sim 1.50 \times 10^8\,\text{km}$ you have, for a low energy neutrino:

$$\Delta_{ij} = 1.27 \frac{\Delta m^2(\text{eV}^2) L(\text{km})}{E_\nu(\text{GeV})}$$

$$\Delta_{31} = \frac{\Delta m_{31}^2 L}{4E} = \mathcal{O}(10^9) \gg 1,$$

$$\Delta_{21} = \frac{\Delta m_{21}^2 L}{4E} = \mathcal{O}(10^7) \gg 1,$$

where $E = 267\,\text{keV}$ ($267 \times 10^{-6}\,\text{GeV}$), $\Delta m_{32}^2 \sim \Delta m_{31}^2 = 2.457 \times 10^{-3}\,\text{eV}^2$, and $\Delta m_{21}^2 = 7.5 \times 10^{-5}\,\text{eV}^2$.

That tells you that an oscillation experiment is out of the question with solar neutrinos. The required energy resolution (see §2.3 [28]) would be

$$\Delta E_\nu \approx \frac{E_\nu}{N_{osc}} \tag{11}$$

For the distance from the Sun to Earth, $L \sim 1.50 \times 10^8\,\text{km}$, that implies $N_{osc} \approx L/\lambda_{osc} \simeq 17 \times 10^6$ oscillations of the $267\,\text{keV}$ neutrino with $\lambda_{osc} \sim 8.8\,\text{km}$ we are considering (setting aside other factors like whether the mass states of the neutrino packets would have separated over that distance and the averaging that occurs over the region of neutrino production in the Sun, estimated to be $\Delta R \simeq (0.04\text{-}0.20) R_\odot$ or $\simeq 10^5\,\text{km}$).

Using the $\Delta E_\nu$ equation above Eq. (11), that would require a detector with $\mathcal{O}(0.02\,\text{eV})$ resolution. For comparison, the Borexino experiment, using a liquid scintillator detector, has a neutrino energy threshold $E_{min} \sim 150\,\text{keV}$, roughly seven orders of magnitude from the required resolution to detect oscillation as a function of neutrino energy in this case.

[3] The first term in Eq. (8), $\sin^2 2\theta$, is the amplitude term with the mixing angle $\theta$. It determines how much of the peak from the oscillating term is observable .

### B. what about the solar mixing angle and mass squared splitting

Yes, you say, but why are $\theta_{12}$ and $\Delta m^2_{12}$ associated with solar neutrino flavor change when the general equations involve three mixing angles, three square mass splittings?

The $\nu_e$ electron neutrino vacuum mix (probabilities, hence square the mixing matrix elements; see Eq. (21) $U$ mixing matrix farther below) of mass states $\nu_j$ is:

$$\begin{aligned}|\nu_e\rangle &= U^2_{e1}|\nu_1\rangle + U^2_{e2}|\nu_2\rangle + U^2_{e3}|\nu_3\rangle \\ &= [\cos\theta_{12}\cos\theta_{13}]^2|\nu_1\rangle + [\sin\theta_{12}\cos\theta_{13}]^2|\nu_2\rangle + [\sin\theta_{13}|]^2\nu_3\rangle \\ &= 0.6805|\nu_1\rangle + 0.2977|\nu_2\rangle + 0.0218|\nu_3\rangle \qquad \text{for } \theta_{12} = 33.48°,\ \theta_{13} = 8.50° \end{aligned} \tag{12}$$

The electron neutrino is therefore 68% $\nu_1$ and $\sim$ 30% $\nu_2$ (almost no $\nu_3$ at 2%), so it is natural to associate the $\theta_{12}$ angle that specifies the $\nu_1$ and $\nu_2$ mixing with the solar electron neutrino, and the mass split difference between them, $\Delta m^2_{12}$.

### C. the low energy region (LER)

As we mentioned above 14, there are two regimes in the Borexino graph Fig. 1 of experimental observations of solar $\nu_e$ survival fractions (three if you count the transition region between the two, the predicted upturn from the low $P_{ee} \approx 0.3$ around 10 MeV neutrino energy, to nearer $P_{ee} \gtrsim 0.4$ at 5 MeV as the MSW effect begins to turn off, apparently reported for the first time by Super-Kamiokande at Neutrino2020).

We detect about 0.55 of the expected low energy (expectation $\langle E_\nu\rangle \sim 267\,\text{keV}$) electron neutrinos from $pp$ fusion and subsequent $\beta^+$-decay:

$$p + p \to {}^2_1\text{H} + e^+ + \nu_e$$

(${}^2_1$H is deuterium, i.e., one proton + one neutron, resulting from $\beta^+$ decay of the initial diproton $p+p$ fusion.)

For these lower energy solar neutrinos there is not much matter potential. Using a PDG formula for the electron number density $N_e\,\text{cm}^{-3}$ required for MSW resonance[4], we can

[4] Equation (14.60) [24].

quantify this:

$$N_e^{res} = \frac{\Delta m_{jk}^2 \cos 2\theta_{jk}}{2E_\nu \sqrt{2} G_F} \simeq 6.56\ \times 10^6\ \frac{\Delta m_{jk}^2 [\mathrm{eV}^2]}{E_\nu [\mathrm{MeV}]}\ \cos 2\theta_{jk}\, \mathrm{cm}^{-3}\, \mathrm{N_A} \tag{13}$$

where $\mathrm{N_A} = 6.022140857\ \times 10^{23}$ is Avogadro's number and $G_F = 1.1664\ \times 10^{-5}\,\mathrm{GeV}^{-2}$ is the Fermi coupling constant.

For the $\Delta m_{12}^2 = 7.5\ \times 10^{-5}\,\mathrm{eV}^2$ mass split and associated mixing angle $\theta_{12} = 33.48°$ (we are using 2016 global fit values [32] to maintain closer correspondence with the 2014 Borexino graph), a *pp* neutrino with $E_\nu = 0.267\,\mathrm{MeV}$ would require an electron number density of

$$\begin{aligned} N_e^{res} &\simeq 6.56\ \times 10^6\ \frac{7.5\ \times 10^{-5}\ [\mathrm{eV}^2]}{0.267\ [\mathrm{MeV}]}\ \cos[(2)(0.5843\,\mathrm{rad})]\,\mathrm{cm}^{-3}\, 6.022140857\ \times 10^{23} \\ &= 4.3431\ \times 10^{26}\,\mathrm{cm}^{-3} \end{aligned}$$

Using the above formula Eq. (13), we find that a $E_\nu = 0.267\,\mathrm{MeV}$ neutrino would require $N_e(t = 0) = 4.3431\ \times 10^{26}\,\mathrm{cm}^{-3}\ e^-$ number density at birth for MSW resonance (our criterion here for significant matter effect). Consulting the B16(GS98) SSM (Standard Solar Model) data [33] we find the radial location of the maximum emission point of the *pp* neutrinos is $r_\odot = 0.0990$ ($r_\odot = 0$ is the center of the Sun, $r_\odot = 1$ the normalized radius) and the electron number density there is $N_e(r_\odot = 0.0990) = 4.0249\ \times 10^{25}\,\mathrm{cm}^{-3}$. Therefore, for significant matter effect on the *pp* neutrino flavor evolution we require approximately an order of magnitude more electron number density than is present at the production location in the solar core.

### D. can LER survival probability depend on only the $\theta_{12}$ angle?

Since there is almost no MSW effect at low energy, *pp* neutrino flavor oscillations occur in the Sun as in vacuum (no significant level jump probability either, but we will not get into that here). Now, since we cannot observe solar neutrino oscillations at Earth (for the reasons discussed in Section V A), we may only observe average flavor change. For the solar *pp* $\nu_e$, PDG offers a *two-neutrino model* survival probability equation ((14.89) in [24]):

$$\bar{P}^{2\nu}(\nu_e \to \nu_e) \simeq 1 - \frac{1}{2} \sin^2 2\theta_{12} \tag{14}$$

We obtain $\bar{P}^{2\nu}(\nu_e \to \nu_e) \simeq 0.5766$ for $\theta_{12} = 33.48°$ with this equation. Aha! We see the low energy solar neutrino survival probability (the left side of the Borexino graph Fig. 1 ),

$P_{ee} \approx 0.55$) is dependent solely on the $\theta_{12}$ mixing angle. Why is this true when there are three mass splittings and three mixing angles?

### E. Incoherent vs coherent detection, some quantum mechanics

The amplitude for finding a $\nu_{l'}$ in an original $\nu_l$ neutrino flux is[5]

$$\begin{aligned}\langle \nu_{l'} \mid \nu_l \rangle &= \sum_j \langle \nu_{j,l'} | U^\dagger_{jl'} e^{-iE_j t} U_{jl} | \nu_{j,l} \rangle \\ &= \sum_j e^{-iE_j t} U_{lj} U^*_{l'j}\end{aligned} \tag{15}$$

where we used the orthogonality relation $\langle \nu_j \mid \nu_k \rangle = \delta_{jk}$.

If the experimental conditions permit coherent detection, then in usual quantum mechanical fashion you take the squared modulus of the transition amplitude Eq. (15) to obtain the probability for the transition:

$$P_{\nu_l \to \nu_{l'}}(t) = |\langle \nu_{l'} \mid \nu_l(t) \rangle|^2 = \left| \sum_j e^{-iE_j t} U_{lj} U^*_{l'j} \right|^2 \tag{16}$$

The three-neutrino oscillation equation is derived from that equation Eq. (16)[6]. However, since we can detect only an average, we cannot sum amplitudes, but instead must sum probabilities[7], i.e., sum the squared amplitudes:

$$\bar{P}(\nu_l \to \nu_{l'}) = \sum_j \left| \langle \nu'_l \mid \nu_j \rangle e^{-iE_j t} \langle \nu_j \mid \nu_l \rangle \right|^2 = \sum_j |U_{l'j}|^2 |U_{lj}|^2 \tag{17}$$

In Eq. (17) the $\bar{P}$ indicates the average probability (this is the result in equation (14.50) from §14.7 [24]). Notice the difference between Eq. (16), where the amplitudes are summed inside the squared modulus, and Eq. (17), where the squared modulus of the product of the production braket $\langle \nu_j \mid \nu_l \rangle$ and detection braket $\langle \nu'_l \mid \nu_j \rangle$ (the probability) is summed.

Note that the exponential phase evolution factor vanishes in a squared modulus, i.e.,

$$\left| e^{-iE_j t} \right|^2 = \left( \sqrt{e^{-iE_j t} \; e^{+iE_j t}} \right)^2 = \left( \sqrt{e^{E_j t - iE_j t}} \right)^2 = \left( \sqrt{e^0} \right)^2 = 1 \tag{18}$$

[5] See, for example, equation (37) in [31].

[6] See appendix Section VII) for details of that derivation.

[7] If quantum mechanics is unfamiliar to you, transcripts of Richard Feynman's 1963 and 1964 lectures on quantum mechanics at Caltech are available at [34], as are his introductory lectures on physics generally. Barton Zwiebach's quantum physics lectures at MIT are useful also [35]

Then we take the squared moduli of the $U$ transition elements in Eq. (17),

$$|U_{l'j}|^2|U_{lj}|^2 = \left(\sqrt{U_{l'j}U^*_{l'j}}\right)^2 \left(\sqrt{U_{lj}U^*_{lj}}\right)^2 = U_{l'j}U^*_{l'j}U_{lj}U^*_{lj} \tag{19}$$

For $l = l' = e$, Eq. (17) expands to

$$\begin{aligned}\bar{P}(\nu_e \to \nu_e) &= |U_{e1}U^*_{e1}|^2 + |U_{e2}U^*_{e2}|^2 + |U_{e3}U^*_{e3}|^2 \\ &= U_{e1}U^*_{e1}U_{e1}U^*_{e1} + U_{e2}U^*_{e2}U_{e2}U^*_{e2} + U_{e3}U^*_{e3}U_{e3}U^*_{e3}\end{aligned} \tag{20}$$

Replace the $U_{\alpha j}$ in Eq. (20) with their values from the standard parameterization of the PMNS matrix:

$$\begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix} = \begin{pmatrix}U_{e1} & U_{e2} & U_{e3}\\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3}\\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3}\end{pmatrix}\begin{pmatrix}\nu_1\\ \nu_2\\ \nu_3\end{pmatrix}$$

$$\begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix} = \begin{pmatrix}c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{CP}}\\ -s_{12}c_{23} - c_{12}s_{13}s_{23}e^{i\delta_{CP}} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta_{CP}} & c_{13}s_{23}\\ s_{12}s_{23} - c_{12}s_{13}c_{23}e^{i\delta_{CP}} & -c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\delta_{CP}} & c_{13}c_{23}\end{pmatrix}\begin{pmatrix}\nu_1\\ \nu_2\\ \nu_3\end{pmatrix} \tag{21}$$

$c_{12}$ denotes $\cos\theta_{12}$, $s_{13}$ is $\sin\theta_{13}$ and so on. We are not interested in the possibility of CP violation in the solar context, so just let $\delta_{CP} = 0$.

We obtain:

$$\begin{aligned}\bar{P}(\nu_e \to \nu_e) &= U_{e1}U^*_{e1}U_{e1}U^*_{e1} + U_{e2}U^*_{e2}U_{e2}U^*_{e2} + U_{e3}U^*_{e3}U_{e3}U^*_{e3} \\ &= \cos^4(\theta_{12})\cos^4(\theta_{13}) + \sin^4(\theta_{12})\cos^4(\theta_{13}) + \sin^4(\theta_{13})\end{aligned} \tag{22}$$

Eq. (22) simplifies to

$$\begin{aligned}\bar{P}(\nu_e \to \nu_e) &= \cos^4\theta_{13}\left[\cos^4\theta_{12} + \sin^4\theta_{12}\right] + \sin^4(\theta_{13}) \\ &= \cos^4\theta_{13}\left[1 - \frac{1}{2}\sin^2 2\theta_{12}\right] + \sin^4(\theta_{13})\end{aligned} \tag{23}$$

where we made the substitution:

$$\left[\cos^4(\theta_{12}) + \sin^4\theta_{12}\right] \Longrightarrow \left[1 - \frac{1}{2}\sin^2 2\theta_{12}\right]$$

With $\theta_{13} = 8.5°$, $\sin^4(\theta_{13}) = 0.00048$ and may reasonably be dropped from Eq. (23). With that value of mixing angle, $\cos^4\theta_{13} = 0.96 \approx 1$ and may be omitted also in Eq. (23), leading to the PDG two-neutrino low energy solar neutrino survival probability we introduced as Eq. (14) above:

$$\bar{P}^{2\nu}(\nu_e \to \nu_e) = \underbrace{\cos^4\theta_{13}}_{\sim 1}\left[1 - \frac{1}{2}\sin^2 2\theta_{12}\right] + \underbrace{\sin^4(\theta_{13})}_{\approx 0}$$
$$\simeq 1 - \frac{1}{2}\sin^2 2\theta_{12} \qquad (24)$$

### F. conclude small mixing of $\nu_3$ in $\nu_e$ explains dominant $\nu_1, \nu_2$

So, for the LER low energy range solar neutrinos, we may reasonably suggest that the reason the $\theta_{12}$ mixing angle can approximately account for the observed survival probability $\approx 0.55$ is that the $\theta_{13}$ mixing angle is very small (which means that the fraction of $\nu_3$ mass state is very small in the electron neutrino, since $U_{e3} = \sin\theta_{13}$, referring to the $U$ matrix Eq. (21) above) and so drops out of the averaged (incoherent) survival probability equation Eq. (24).

### G. the high energy region (HER)

On the right side of the Borexino graph above Fig. 1, we see the survival probability $P_{ee}$ observed by various experiments is $\sim 0.3$ in the 10 MeV area (this is the sweet spot for the SNO experiment, though their threshold is lower, $\sim 5$ MeV). The solar neutrinos in this range are predominantly produced in the $\beta^+$ decay ${}^8_5\mathrm{B} \to {}^8_4\mathrm{Be}^* + e^+ + \nu_e$ with energy expectation $\langle E_\nu \rangle = 6.735$ MeV.

This 0.3 survival probability is very close to simply $\sin^2\theta_{12} = 0.304$ for $\theta_{12} = 33.48°$. Why is $\theta_{12}$ again so closely connected to an observed survival probability of solar neutrinos?

The $\Delta m^2_{12}$ oscillation length for a 10 MeV neutrino is $\sim 330$ km (about $\sim 10$ km for the $\Delta m^2_{13}$ split), again much smaller than the distance from the Sun to Earth, so detecting oscillations would again be impossible on that basis alone (and the averaging over the region of neutrino production in the Sun, $\Delta R \simeq (0.04\text{-}0.20)R_\odot \simeq 10^5$ km as we mentioned earlier). However, there is a much more significant feature for a 10 MeV solar neutrino, i.e., the matter potential.

### H. matter matters

Consulting the B16(GS98) SSM (Standard Solar Model) again [33], we find that the $^8$B neutrino production peaks around $r = 0.0460\, r_\odot$ normalized solar radius with electron number density at that location $N_e \sim 5.467 \times 10^{25}\,\mathrm{cm}^{-3}$. Using the PDG $N_e^{res}$ equation above Eq. (13), we see that a 10 MeV neutrino reaches MSW resonance for the $\Delta m^2_{12}$ mass split at $N_e^{res} = 1.1596 \times 10^{25}\mathrm{cm}^{-3}$. The $^8$B solar neutrino is produced at almost 5 times the required electron number density for MSW resonance, so we expect the matter potential to be highly significant.

For the $\Delta m^2_{13} = 2.457 \times 10^{-3}\,\mathrm{eV}^2$ mass split, on the other hand, the required electron density for resonance, $N_e^{res} = 9.2823 \times 10^{26}\mathrm{cm}^{-3}$, is more than ten times the available electron number density. We understand, then, why $\Delta m^2_{13}$ is not significantly associated with solar neutrino flavor change in the high energy MSW region.

Consider a 10 MeV $^8$B electron neutrino weak (flavor) state $|\nu_e\rangle$ produced at $t = 0$. Read off the $\nu_e$ weak state description in terms of mass states from the top row $(U_{e1}, U_{e2}, U_{e3})$ of $U$ in Eq. (21), applied to the mass state column on the rhs, $(\nu_1, \nu_2, \nu_3)^T$ (this notation means to write the elements of this row as a column, i.e., $T$ transpose, since is typographically inconvenient to include a column inline):

$$\begin{aligned} |\nu_e\rangle &= U_{e1}|\nu_{1m}\rangle + U_{e2}|\nu_{2m}\rangle + U_{e3}|\nu_{3m}\rangle \\ &= \cos\theta_{12m}\cos\theta_{13m}|\nu_{1m}\rangle + \sin\theta_{12m}\cos\theta_{13m}|\nu_{2m}\rangle + \sin\theta_{13m}|\nu_{3m}\rangle \end{aligned} \tag{25}$$

If you are not familiar with the procedure (matrix vector multiplication) for assigning the $U$ elements to mass states for a particular flavor state as we did in Eq. (25) (we are using the same mixing matrix, we have simply transformed the parameters under the influence of the matter potential, denoted with $_m$ subscripts), informally put, think of $\nu_e$ in row 1of the column vector on the lhs in Eq. (21) "owning" row 1 of the $U$ matrix on the rhs. To see the composition of $\nu_e$ in terms of the $\nu_1, \nu_2, \nu_3$ mass states on the rhs, just pick up that row 1 from the $U$ matrix and place it as a column to the left of the $\nu_1, \nu_2, \nu_3$ column like this:

$$\begin{pmatrix} U_{e1} \\ U_{e2} \\ U_{e3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}$$

Then read off each row of the two columns as a product, placing a plus sign between each row so constructed:

$$\nu_e = U_{e1}\nu_1 + U_{e2}\nu_2 + U_{e3}\nu_3$$

Similarly, $\nu_\mu$ owns row 2 of $U$, so (if you were interested in $\nu_\mu$ ) you would pick up that second row of $U$, placing it to the left of the mass state column and then read off the components as we did with the $\nu_e$.

Again, the $_m$ subscripts indicate that these are states and values in solar plasma, i.e., modified by matter effect (increased $\nu_e$ coherent forward elastic scattering potential off electrons, i.e., from exchange of a virtual $W$-boson, for $\nu_e$ preferentially in relation to $\nu_{\mu,\tau}$). The $\theta_{12}$ mixing angle with matter effect is (we are following §2.3 [28] here, who included the small correction from the $\nu_1\nu_3$ mixing):

$$\cos 2\theta_{12}^m = \frac{\cos 2\theta_{12} - \cos^2\theta_{13}\epsilon_{12}}{\sqrt{\left(\cos 2\theta_{12} - \cos^2\theta_{13}\epsilon_{12}\right)^2 + \sin^2 2\theta_{12}}} \tag{26}$$

where

$$V_e = \sqrt{2}G_F N_e \tag{27}$$

and

$$\epsilon_{12} \equiv \frac{2V_e E}{\Delta m_{12}^2} \tag{28}$$

The matter potential $V_e$ is calculated with Eq. (27) using our test case $^8$B neutrino with specified electron density $N_e$ and energy $E$:

$$V_e = \sqrt{2}G_F N_e(\hbar c)^3 = [\sqrt{2}][1.1664\ \times 10^{-5}\,\mathrm{GeV}^{-2}][5.467\ \times 10^{25}\,\mathrm{cm}^{-3}][1.97\ \times 10^{-14}\,\mathrm{GeV\,cm}]^3$$

We obtain $V_e = 6.93\ \times 10^{-21}\,\mathrm{GeV}$ (or $V_e = 6.93\ \times 10^{-12}\,\mathrm{eV}$ if you are comparing with the general potential in solar core given in the literature, i.e., $\mathcal{O}(10^{-12}\,\mathrm{eV})$, see, for example, pg 20 [31]).

$\epsilon_{12}$ is then (you see factors converting all energy units to GeV)

$$\epsilon_{12} = \frac{2V_e E}{\Delta m_{12}^2} = \frac{(2)(6.93\ \times 10^{-21}\,\mathrm{GeV})(10\ \times 10^{-3}\mathrm{GeV})}{(7.5\ \times 10^{-5}\ \times 10^{-18}\mathrm{GeV}^2)} = 1.848$$

We see the energy units cancel, making $\epsilon_{12}$ a dimensionless ratio of the matter potential of the 10 MeV neutrino in $5.467\ \times 10^{25}\,\mathrm{cm}^{-3}$ electron number density to the $\Delta m_{12}^2$ mass splitting, 1.848 indicating that the matter potential is nearly twice the mass splitting at production.

Plug that value of $\epsilon_{12}$ into Eq. (26) and obtain the cosine of $2\theta_{12m}$ mixing angle in matter:

$$\begin{aligned}\cos 2\theta_{12}^m &= \frac{\cos 2\theta_{12} - \cos^2\theta_{13}\epsilon_{12}}{\sqrt{(\cos 2\theta_{12} - \cos^2\theta_{13}\epsilon_{12})^2 + \sin^2 2\theta_{12}}} \\ &= \frac{(0.3914\,\mathrm{rad}) - (0.9782\,\mathrm{rad})(1.848)}{\sqrt{[(0.3914\,\mathrm{rad}) - (0.9782\,\mathrm{rad})(1.848)]^2 + (0.8468\,\mathrm{rad})}} \\ &= -0.83854\end{aligned}$$

The arccos of $-0.83854$ is 2.56540 rad. Divide by 2 to obtain the $\theta_{12m}$ mixing angle with the calculated matter potential, 1.28270 rad, or $\sim 74°$. That is significant rotation (positive, counterclockwise) from the vacuum $\theta_{12}$ angle of 33.48°.

We obtain the $\theta_{13m}$ mixing angle with [28]

$$\sin^2\theta_{13}^m = \sin^2\theta_{13}(1 + 2\epsilon_{13}) + \mathcal{O}(s_{13}^2\epsilon_{13}^2, s_{13}^4, \epsilon_{13}) \tag{29}$$

We already have the $V_e$ potential and merely require $\epsilon_{13}$:

$$\epsilon_{13} \equiv \frac{2V_eE}{\Delta m_{13}^2} \tag{30}$$

Inserting in Eq. (30) the previously calculated values, with $\Delta m_{13}^2 = 2.457 \times 10^{-3}\,\mathrm{eV}^2$ the only change from our previous $\epsilon_{12}$ calculation:

$$\epsilon_{13} \equiv \frac{2V_eE}{\Delta m_{13}^2} = \frac{(2)(6.93 \times 10^{-21}\,\mathrm{GeV})(10 \times 10^{-3}\mathrm{GeV})}{(2.457 \times 10^{-3} \times 10^{-18}\mathrm{GeV}^2)} = 0.0564$$

$\theta_{13}^m$ is then

$$\arcsin(\sqrt{0.024259}) = 0.1566\,\mathrm{rad}$$

or 8.9704°. As expected, the $\theta_{13}^m$ mix angle has not changed much from the vacuum value 8.5°, the larger $\Delta m_{13}^2$ mass split requiring more electron density than is available in order to produce significant rotation.

Referring to Eq. (25) above, we can look at the mass state mix at production for our test neutrino (equation repeated here for convenience):

$$\begin{aligned}|\nu_e\rangle &= U_{e1}|\nu_{1m}\rangle + U_{e2}|\nu_{2m}\rangle + U_{e3}|\nu_{3m}\rangle \\ &= \cos\theta_{12m}\cos\theta_{13m}|\nu_{1m}\rangle + \sin\theta_{12m}\cos\theta_{13m}|\nu_{2m}\rangle + \sin\theta_{13m}|\nu_{3m}\rangle\end{aligned}$$

The square of the mixing elements $U_{e1m}, U_{e1m}, U_{e1m}$ in Eq. (21) with mixing angle in matter $\theta_{jk}^m$ give the probability fraction for each mass state in the electron neutrino produced under the specified conditions in the solar plasma (the crowded ket below $|\nu_e(N_e, E, t=0)\rangle$ is intended to remind you that the state of this neutrino at production $t=0$ is dependent on its energy and the local electron density in solar plasma):

$$
\begin{gathered}
|\nu_e(N_e, E, t=0)\rangle = \\
\{\, [\cos\theta_{12m}\cos\theta_{13m}]^2 = [\cos(74^\circ)\cos(8.9704^\circ)]^2 = 0.0788|\nu_{1m}\rangle \,\} + \\
\{\, [\sin\theta_{12m}\cos\theta_{13m}]^2 = [\sin(74^\circ)\cos(8.9704^\circ)]^2 = 0.8969|\nu_{2m}\rangle \,\} + \quad \Longleftarrow \\
\{\, [\sin\theta_{13m}]^2 = [\sin(8.9704^\circ)]^2 = 0.0243|\nu_{3m}\rangle \,\}
\end{gathered}
$$

We see the 10 MeV $^8$B electron neutrino consists almost entirely $(0.8969 \sim 90\%))$ of the $|\nu_{2m}\rangle$ mass state.[8] Because the solar plasma electron number density decreases smoothly (approximately as an exponential), the $|\nu_{2m}\rangle$ evolves continuously (without level jumps) into the state $|\nu_2\rangle$ at the surface of the Sun and is ultimately detected at Earth as that $|\nu_2\rangle$ mass state, with the vacuum flavor probabilities assigned to $\nu_2$ ([24] §14.8.2.1, [17] § III.D).

To see the probabilities for detection of $|\nu_e\rangle$ on interaction with the incoming lone $^8$B $|\nu_2\rangle$ mass state , we can dagger † the $U$ mixing matrix Eq. (21) (swap rows and columns, conjugating any complex elements, but we are setting $\delta_{CP}=0$, i.e., considering $U$ real, so there will not be any actual complex conjugation),

$$
\begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} = \underbrace{\begin{pmatrix} U_{e1}^* & U_{\mu 1}^* & U_{\tau 1}^* \\ U_{e2}^* & U_{\mu 2}^* & U_{\tau 2}^* \\ U_{e3}^* & U_{\mu 3}^* & U_{\tau 3}^* \end{pmatrix}}_{U^\dagger} \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} \tag{31}
$$

The elements of $U^\dagger$ Eq. (31) above are:

$$
U^\dagger = \begin{pmatrix} c_{12}c_{13} & -s_{12}c_{23} - c_{12}s_{13}s_{23}e^{-i\delta_{CP}} & s_{12}s_{23} - c_{12}s_{13}c_{23}e^{-i\delta_{CP}} \\ s_{12}c_{13} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{-i\delta_{CP}} & -c_{12}s_{23} - s_{12}s_{13}c_{23}e^{-i\delta_{CP}} \\ s_{13}e^{i\delta_{CP}} & c_{13}s_{23} & c_{13}c_{23} \end{pmatrix} \tag{32}
$$

Use the same method we discussed earlier 23 to associate the flavor states $\nu_e, \nu_\mu, \nu_\tau$ in the rhs column with the $U^\dagger$ row "owned" by each of the mass states $\nu_1, \nu_2, \nu_3$ in the column

[8] A 2006 study [36] calculated the integrated weighted fraction of $\nu_2$ over threshold 5.5 MeV recoil electron kinetic energy as $91 \pm 2\%$.

on the lhs, we find the flavor composition for the $|\nu_2\rangle$ mass state is

$$|\nu_2\rangle = U^*_{e2}\nu_e + U^*_{\mu 2}\nu_\mu + U^*_{\tau 2}\nu_\tau$$

We are back in vacuum conditions on Earth, at least as far as a solar neutrino is concerned, so read off and square the $U^\dagger$ elements Eq. (31) (Eq. (32)) using the vacuum mixing angles, $\theta_{12} = 33.48°$ and $\theta_{13} = 8.50°$:

$$\begin{gathered}
|\nu_2\rangle = |U^*_{e2}|^2|\nu_e\rangle + |U^*_{\mu 2}|^2|\nu_\mu\rangle + |U^*_{\tau 2}|^2|\nu_\tau\rangle \\
= [\sin\theta_{12}\cos\theta_{13}]^2\,|\nu_e\rangle + \\
[-\sin\theta_{12}\sin\theta_{13}\sin\theta_{23} + \cos\theta_{12}\cos\theta_{23}]^2\;|\nu_\mu\rangle + \\
[-\sin\theta_{12}\sin\theta_{13}\cos\theta_{23} - \sin\theta_{23}\cos\theta_{12}]^2\,|\nu_\tau\rangle \\
= \underbrace{0.2977|\nu_e\rangle}_{\text{cf } P_{ee}\sim 0.3} + 0.3159|\nu_\mu\rangle + 0.3865|\nu_\tau\rangle
\end{gathered} \tag{33}$$

We find the single $\nu_2$ state detected at Earth produces closely the observed survival probability $P_{ee} \sim 0.3$ of $^8$B electron neutrinos on the right side of the Borexino graph above Fig. 1. With the $\cos^2\theta_{13} = 0.9781 \approx 1$ factor on the $|\nu_e\rangle$ ket in Eq. (33) , this survival probability is produced largely by simply $\sin^2\theta_{12} = 0.304$.

### I. single equation, $\theta_{12}$, $\Delta m^2_{12}$ approximates solar $\nu_e$ survival probability

It was easier to address the connection between $\theta_{12}$ and $\Delta m^2_{12}$ and solar neutrino flavor change separately in the low and high energy regimes, but it is possible to write a single equation that predicts the survival probability over the entire range in the Borexino graph above Fig. 1.

As a pedagogical tool, now that we know we must sum probabilities rather than amplitudes when coherent detection is impossible (as we discussed in Section V E)), and that $\theta_{13}$ has a small impact on the detection probability (Section V F), Eq. (33)), we can reduce to a two-neutrino model (set $\theta_{13} = 0$ and use the first row of $U$ Eq. (21), producing the two-neutrino matrix shown in Eq. (2) ) and write for the neutrino ket at production in the solar plasma:

$$\begin{aligned}|\nu_e(t=0,N_e,E)\rangle &= \sum_{k=1}^{2} U^m_{ek}|\nu_k\rangle \\ &= U^m_{e1}|\nu_1\rangle + U^m_{e2}|\nu_2\rangle \\ &= (\cos\theta^m_{12})|\nu_1\rangle + (\sin\theta^m_{12})|\nu_2\rangle\end{aligned} \tag{34}$$

Basically, the same mixing relations apply, but we use the $\theta^m_{12}$ mixing angle transformed by any matter potential that develops at the given density and neutrino production energy.

Then write for detection as $\nu_e$ at Earth:

$$\begin{aligned}|\nu_e\rangle &= \sum_{k=1}^{2} U_{ek}|\nu_k\rangle \\ &= U_{e1}|\nu_1\rangle + U_{e2}|\nu_2\rangle \\ &= (\cos\theta_{12})|\nu_1\rangle + (\sin\theta_{12})|\nu_2\rangle\end{aligned} \tag{35}$$

where here we use the vacuum value of the $\theta_{12}$ mixing angle which applies at detection[9].

We project the production ket onto the detection ket, square and sum to obtain the incoherent probability to detect the $\nu_e$ at Earth:

$$|\langle\nu_e \mid \nu_e(t=0,N_e,E)\rangle|^2 = \sum_{k}^{2}|U_{ek}|^2|U^m_{ek}|^2$$

$$P_{ee} = (\cos\theta_{12})^2(\cos\theta^m_{12})^2 + (\sin\theta_{12})^2(\sin\theta^m_{12})^2 \tag{36}$$

Using an equation for $\theta^m_{12}$ without any $\nu_3$ factors:

$$\cos 2\theta^m_{12} = \frac{\left(\cos 2\theta_{12} - \frac{\Delta V}{\Delta m^2_{12}}\right)}{\sqrt{\left(\cos 2\theta_{12} - \frac{\Delta V}{\Delta m^2_{12}}\right)^2 + \sin^2 2\theta_{12}}} \tag{37}$$

where

$$\Delta V = 2\sqrt{2}G_F E N_e$$

[9] We discovered this technique of taking the product of the moduli squared of the mixing elements at production with matter effect and the moduli squared of the mixing elements with detection conditions at Earth in §2.3 of [28], in particular equation (11).

To avoid having to calculate $\sin\theta_{12}^m$ (or to obtain $\cos\theta_{12}^m$ from $\cos 2\theta_{12}^m$), Eq. (36) can be rewritten as

$$P_{ee} = \frac{1}{2}(1 + \cos 2\theta_{12}^m \cos 2\theta_{12}) \tag{38}$$

Surprisingly, the simplified equation Eq. (38) tracks the 2014 Borexino equation (94) [29] (which uses the $\nu_3$ correction to the $\cos 2\theta_{12}^m$ equation Eq. (26) we presented earlier above as well as a leading $\cos^4\theta_{13}$ adjustment to the probability) fairly well, albeit $\sim 4\%$ higher survival probability throughout:

$$\text{Borexino equation (94):} \quad P_{ee}^{3\nu} = \frac{1}{2}\cos^4\theta_{13}\left(1 + \cos 2\theta_{12}^M \cos 2\theta_{12}\right) \tag{39}$$

We graphed our two-neutrino equation Eq. (38) and the 2014 Borexino equation Eq. (39) in Fig. 2 (cf Fig. 1).

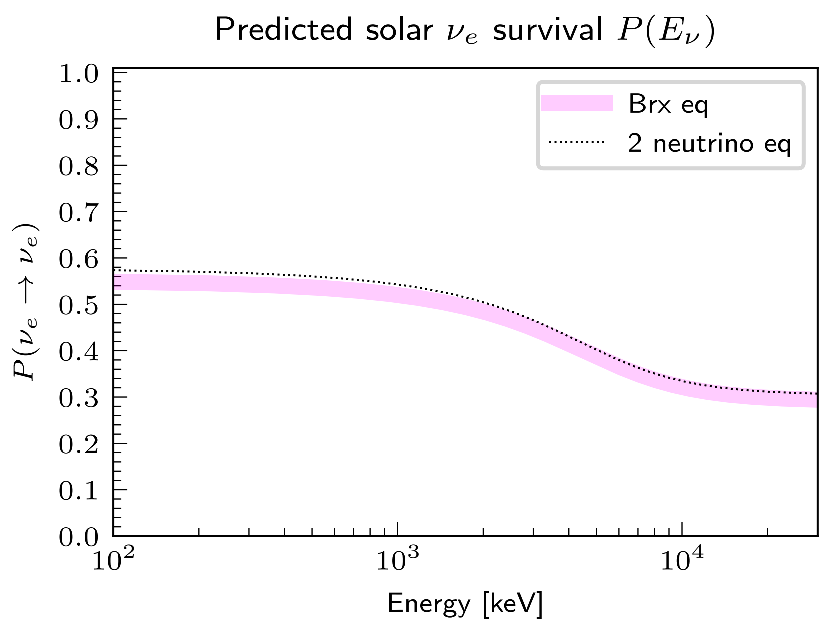


FIG. 2. 2014 Borexino experiment survival probability equation compared to our two-neutrino simplification (see text for context).

## VI. CONCLUSION

We hope we have made some of the calculations involved in solar neutrino flavor change more accessible and that we have answered the question as to why $\theta_{12}$ and $\Delta m_{12}^2$ are associated with solar neutrinos.

## VII. DERIVATION OF THREE-NEUTRINO OSCILLATION EQUATION

We rely on several sources throughout the following derivation: §3.1.1. in [37], *Handout 11: Neutrino Oscillations* in [25], *Appendix A* in [38], §14.7 in [24].

We stipulate coherent production and detection of the neutrino. As we stated in Section V E), we have the amplitude for finding a $\nu_{l'}$ in an original $\nu_l$ neutrino flux:

$$\begin{aligned} \langle \nu_{l'} \mid \nu_l \rangle &= \sum_j \langle \nu_{j,l'} | U^\dagger_{jl'} e^{-i\phi_j} U_{jl} | \nu_{j,l} \rangle \\ &= \sum_j e^{-i\phi_j} U_{lj} U^*_{l'j} \end{aligned} \tag{40}$$

where we used the orthogonality relation $\langle \nu_j \mid \nu_k \rangle = \delta_{jk}$. The exponential evolution factor was created as a plane wave solution in relativistic quantum mechanics as shown earlier in Eq. (4) and Eq. (5). Recall from that discussion 9 $\phi_j = x^\mu p_\mu = Et - \vec{p} \cdot \vec{x} = Et - \vec{p}L$ in the argument of the exponential.

Because we have specified coherent production and detection, we take the squared modulus of the transition amplitude Eq. (40) to obtain the probability for the transition, acknowledging the particular case of three neutrinos:

$$P_{\nu_l \to \nu_{l'}} = |\langle \nu_{l'} \mid \nu_l(t) \rangle|^2 = \left| \sum_j^3 e^{-i\phi_j} U_{lj} U^*_{l'j} \right|^2 \tag{41}$$

We have a complex identity to assist us here. If you laboriously work out the implicit general multivariate complex polynomial product $|z_1+z_2+z_3|^2 = (z_1+z_2+z_3)(z_1+z_2+z_3)^*$ (the $^*$ indicates the conjugate of the expression) and consolidate the terms using complex identities you can obtain

$$|z_1 + z_2 + z_3|^2 \equiv |z_1|^2 + |z_1|^2 + |z_3|^2 + 2\Re(z_1 z_2^* + z_1 z_3^* + z_2 z_3^*) \tag{42}$$

Eq. (42) is usually just offered as an identity, but we were curious where it originated, so executed the complex product indicated with a CAS (computer algebra system, `SymPy` to be specific) and spent some time arriving at the suggested identity. We could not find any particular label for this identity (hence the mouthful we proposed above "general multivariate complex polynomial product"). $z_j \in \mathbb{C}$ is a complex quantity, canonically of form

$z = (x, y) = x \pm iy$, and the notation $z_j^*$ indicates the complex conjugate of $z_j$ (toggle the sign on the imaginary term). The modulus or absolute value of the complex quantity is $|z_j| = \sqrt{z_j z_j^*}$. We intend by the notation $2\Re(zw^*)$ that for $z, w \in \mathbb{C}$:

$$\begin{aligned} |z+w|^2 &= (z+w)(z+w)^* \\ &= zz^* + \underbrace{zw^* + z^*w}_{\text{replace below as } 2\Re(zw^*)} + ww^* \\ &= \underbrace{|z|^2}_{zz^*} + 2\Re(zw^*) + \underbrace{|w|^2}_{ww^*} \end{aligned} \tag{43}$$

Preparing to apply the identity Eq. (42), let us define:

$$\begin{aligned} z_1 &= e^{-i\phi_i} U_{li} U_{l'i}^* \\ z_2 &= e^{-i\phi_j} U_{lj} U_{l'j}^* \\ z_3 &= e^{-i\phi_k} U_{lk} U_{l'k}^* \end{aligned} \tag{44}$$

Note that we can omit the exponential term on any expression where the squared modulus of $z_1, z_2, z_3$ arises since $-i\phi + i\phi = 0$ and the exponential becomes simply 1. For example,

$$\begin{aligned} z_j &= U_{lj} U_{l'j}^* e^{-i\phi_j} \\ |z_j|^2 &= \left(\sqrt{z_j z_j^*}\right)^2 = \left(U_{lj} U_{l'j}^* e^{-i\phi_j}\right)\left(U_{lj}^* U_{l'j} e^{+i\phi_j}\right) \\ &= \left(U_{lj}^* U_{l'j} U_{lj} U_{l'j}^*\right)\left(e^{i\phi_j - i\phi_j}\right) = \left|U_{lj}^* U_{l'j}\right|^2 (e^0) \\ &= \left|U_{lj}^* U_{l'j}\right|^2 \end{aligned}$$

Substitute the assigned $z_j$ terms from Eq. (44) back into the complex identity Eq. (42) and obtain:

$$\begin{aligned} P_{\nu_l \to \nu_{l'}} &= |U_{li} U_{l'i}^*|^2 + |U_{lj} U_{l'j}^*|^2 + |U_{lk} U_{l'k}^*|^2 + \\ & 2\Re \left\{ U_{li} U_{l'i}^* U_{lj} U_{l'j}^* e^{-i(\phi_i - \phi_j)} \right. \\ & + U_{li} U_{l'i}^* U_{lk} U_{l'k}^* e^{-i(\phi_i - \phi_k)} \\ & \left. + U_{lj} U_{l'j}^* U_{lk} U_{l'k}^* e^{-i(\phi_j - \phi_k)} \right\} \end{aligned} \tag{45}$$

Now we consider the probability $P_{\nu_l \to \nu_{l'}}$ at $t = 0, x = 0$. The argument to the exponential evolution factor, $\phi_j = x^\mu p_\mu = Et - \vec{p} \cdot \vec{x} = Et - \vec{p}L$ as we discussed earlier 9, is then zero (and the exponential then replaced by simply 1), since the neutrino has just been produced and no time has passed or distance propagated, so phase differences between neutrino mass states cannot have developed yet (we hope this is persuasive since we simply want to rid ourselves of these factors for the moment). We note that because $U$ flavor elements are orthogonal,

$$U_{l1}U^*_{l'1} + U_{l2}U^*_{l'2} + U_{l3}U^*_{l'3} = \delta_{ll'} \tag{46}$$

The squared modulus of Eq. (46) is therefore zero also at $t = 0, x = 0$ for $l \neq l'$ and we can say that the probability of a transition is therefore zero, $P_{\nu_l \to \nu_{l'}} = 0$ at $t = 0, x = 0$, which is a reasonable proposition (e.g., an electron neutrino $\nu_e$ is an electron flavor neutrino at production with probability 1 and another flavor with probability 0, hence the Kronecker delta statement $\delta_{ll'}$). We may then write a version of Eq. (45) at $t = 0$ as:

$$\begin{aligned}
P_{\nu_l \to \nu_{l'}}(t=0) &= \delta_{ll'} \\
\delta_{ll'} &= |U_{li}U^*_{l'i}|^2 + |U_{lj}U^*_{l'j}|^2 + |U_{lk}U^*_{l'k}|^2 + \\
&\quad 2\Re\left\{ U_{li}U^*_{l'i}U_{lj}U^*_{l'j} + U_{li}U^*_{l'i}U_{lk}U^*_{l'k} + U_{lj}U^*_{l'j}U_{lk}U^*_{l'k} \right\} \\
|U_{li}U^*_{l'i}|^2 + |U_{lj}U^*_{l'j}|^2 + |U_{lk}U^*_{l'k}|^2 &= \delta_{ll'} - 2\Re\left\{ U_{li}U^*_{l'i}U_{lj}U^*_{l'j} + U_{li}U^*_{l'i}U_{lk}U^*_{l'k} + U_{lj}U^*_{l'j}U_{lk}U^*_{l'k} \right\}
\end{aligned} \tag{47}$$

Eq. (47) allows us to replace

$$|U_{li}U^*_{l'i}|^2 + |U_{lj}U^*_{l'j}|^2 + |U_{lk}U^*_{l'k}|^2$$

in Eq. (45) with the rhs of Eq. (47) to obtain:

$$\begin{aligned}
P_{\nu_l \to \nu_{l'}} = \delta_{ll'} &- 2\Re\left\{ U_{li}U^*_{l'i}U_{lj}U^*_{l'j} + U_{li}U^*_{l'i}U_{lk}U^*_{l'k} + U_{lj}U^*_{l'j}U_{lk}U^*_{l'k} \right\} + \\
& 2\Re\left\{ U_{li}U^*_{l'i}U_{lj}U^*_{l'j}e^{-i(\phi_i - \phi_j)} \right. \\
& + U_{li}U^*_{l'i}U_{lk}U^*_{l'k}e^{-i(\phi_i - \phi_k)} \\
& \left. + U_{lj}U^*_{l'j}U_{lk}U^*_{l'k}e^{-i(\phi_j - \phi_k)} \right\}
\end{aligned} \tag{48}$$

Now we combine the two expressions with real $U$ expressions in Eq. (48) to obtain:

$$\begin{aligned}
P_{\nu_l \to \nu_{l'}} = \delta_{ll'} + 2\Re &\left\{ U_{li}U^*_{l'i}U_{lj}U^*_{l'j}[e^{-i(\phi_i - \phi_j)} - 1] \right. \\
& + U_{li}U^*_{l'i}U_{lk}U^*_{l'k}[e^{-i(\phi_i - \phi_k)} - 1] \\
& \left. + U_{lj}U^*_{l'j}U_{lk}U^*_{l'k}[e^{-i(\phi_j - \phi_k)} - 1] \right\}
\end{aligned} \tag{49}$$

Note that some choose to write instead $[1 - e^{-i(\phi_i - \phi_k)}]$ above, requiring a minus sign on the $2\Re$ expression instead of the positive sign. The signs in the final form of the equation remain the same (but if you convert the cosine in the transform below to a sine you will change the sign on the $\Re$ portion to negative).

We will transform the exponential expression in brackets in Eq. (49) using Euler's formula:

$$e^{-i\theta} = -i\sin(\theta) + \cos(\theta) \tag{50}$$

$$e^{-i(\phi_j - \phi_k)} = \cos(\phi_j - \phi_k) - i\sin(\phi_j - \phi_k)$$

Applying those two transforms to Eq. (49) we have:

$$\begin{aligned} P_{\nu_l \to \nu_{l'}} = \delta_{ll'} + 2\Re \, \{ \, & U_{li}U^*_{l'i}U_{lj}U^*_{l'j}[\cos(\phi_i - \phi_j) - 1] \\ & + U_{li}U^*_{l'i}U_{lk}U^*_{l'k}[\cos(\phi_i - \phi_k) - 1] \\ & + U_{lj}U^*_{l'j}U_{lk}U^*_{l'k}[\cos(\phi_j - \phi_k) - 1] \, \} \\ & - 2\Im \, \{ \, U_{li}U^*_{l'i}U_{lj}U^*_{l'j}[\sin(\phi_i - \phi_j)] \\ & + U_{li}U^*_{l'i}U_{lk}U^*_{l'k}[\sin(\phi_i - \phi_k)] \\ & + U_{lj}U^*_{l'j}U_{lk}U^*_{l'k}[\sin(\phi_j - \phi_k)] \, \} \end{aligned} \tag{51}$$

Note that the bracketed expression in Eq. (49) above became $\cos(\phi_j - \phi_k) - i\sin(\phi_j - \phi_k) - 1$ and we chose to distribute the $-1$ with the cosine real expression rather than the imaginary group (as is always the case in the literature) in Eq. (51).

Now that we have shown clearly the terms in the steps leading to Eq. (51) , we will consolidate the quartets (the $U_{lj}U^*_{l'j}U_{lk}U^*_{l'k}$ terms) in summations:

$$\begin{aligned} P_{\nu_l \to \nu_{l'}} = \delta_{ll'} + 2\Re \sum_{j>k} & U_{lj}U^*_{l'j}U_{lk}U^*_{l'k}[\cos(\phi_j - \phi_k) - 1] \\ - 2\Im \sum_{j>k} & U_{lj}U^*_{l'j}U_{lk}U^*_{l'k}[\sin(\phi_j - \phi_k)] \end{aligned} \tag{52}$$

Now we replace $(\phi_j - \phi_k)$ with

$$\Delta_{jk} = \frac{\Delta m^2_{jk}}{2E} L$$

in Eq. (52) (derived in detail earlier in Section IV):

$$P_{\nu_l \to \nu_{l'}} = \delta_{ll'} + 2\Re \sum_{j>k} U_{lj} U^*_{l'j} U_{lk} U^*_{l'k} [\cos \Delta_{jk} - 1] - 2\Im \sum_{j>k} U_{lj} U^*_{l'j} U_{lk} U^*_{l'k} [\sin \Delta_{jk}] \tag{53}$$

We can replace $\Delta_{jk}$ in Eq. (53) with the explicit $\frac{\Delta m^2_{jk}}{2E} L$ to obtain a form as it is often presented:

$$P_{\nu_l \to \nu_{l'}} = \delta_{ll'} + 2\Re \sum_{j>k} U_{lj} U^*_{l'j} U_{lk} U^*_{l'k} [\cos \frac{\Delta m^2_{jk}}{2E} L - 1] - 2\Im \sum_{j>k} U_{lj} U^*_{l'j} U_{lk} U^*_{l'k} [\sin \frac{\Delta m^2_{jk}}{2E} L] \tag{54}$$

You also see Eq. (54) written with the cosine converted to a $\sin^2$ term using $1 - \cos(2\theta) \longrightarrow 2\sin^2(\theta)$:

$$P_{\nu_l \to \nu_{l'}} = \delta_{ll'} - 4\Re \sum_{j>k} U_{lj} U^*_{l'j} U_{lk} U^*_{l'k} [\sin^2 \frac{\Delta m^2_{jk}}{4E} L] - 2\Im \sum_{j>k} U_{lj} U^*_{l'j} U_{lk} U^*_{l'k} [\sin \frac{\Delta m^2_{jk}}{2E} L] \tag{55}$$

The sign on the imaginary expression should be negative in either case (and positive for antineutrinos). Occasionally you find this written erroneously with a plus sign in the literature[10]. Correct versions include equation (51) in[31], [37] equation (3.11) (though you may be put off by the use of quartet notation),

[10] In [38] equation ($A$-18) for example.

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